TROPICAL SCHEMES, TROPICAL CYCLES, AND VALUATED MATROIDS

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Abstract. We show that the weights on a tropical variety can be recovered from the tropical scheme structure proposed in [GG16], so there is a well-defined Hilbert-Chow morphism from a tropical scheme to the underlying tropical cycle. For a subscheme of projective space given by a homogeneous ideal I we show that the Giansiracusa tropical scheme structure contains the same information as the set of valuated matroids of the vector spaces I_d for $d \geq 0$. We also give a combinatorial criterion to determine whether a given relation is in the congruence defining the tropical scheme structure.

1. Introduction

The tropicalization of a subvariety Y in the n-dimensional algebraic torus T is a polyhedral complex $\operatorname{trop}(Y)$ that is a "combinatorial shadow" of the original variety. Some invariants of Y, such as the dimension, are encoded in $\operatorname{trop}(Y)$. The complex $\operatorname{trop}(Y)$ comes equipped with positive integer weights on its top-dimensional cells, called multiplicities, that make it into a *tropical cycle*. This extra information encodes information about the intersection theory of compactifications of the original variety Y; see for example [KP11].

In [GG16] the authors propose a notion of *tropical scheme structure* for tropical varieties, which takes the form of a congruence on the semiring of tropical polynomials (see Section 2). When $Y \subset T$ is a subscheme defined by an ideal $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ this congruence is denoted by $\mathcal{T}rop(I)$. In [GG16] the tropical scheme structure is defined in the slightly more general context of $\mathbb{F}_1$-schemes.

In this paper we investigate the relation between these tropical schemes, ideals in the semiring of tropical polynomials, and the theory of valuated matroids introduced by Dress and Wenzel [DW92]. We also show that the tropical cycle of a scheme can be reconstructed from the corresponding congruence.

Our first result is the following.

Theorem 1.1. *Let K be a field with a valuation* $\operatorname{val}\colon K \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$*, and let Y be a subscheme of $T \cong (K^*)^n$ defined by an ideal $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then any of the following three objects determines the others:*

1. *The congruence $\mathcal{T}rop(I)$ on the semiring $S := \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ of tropical Laurent polynomials;*
2. *The ideal* $\operatorname{trop}(I)$ *in S;*
3. *The set of valuated matroids of the vector spaces I_d^h, where $I^h \subset K[x_0, \dots, x_n]$ is the homogenization of the ideal I, and I_d^h is its degree d part.*

When the valuation on K is trivial, this says that the tropical scheme structure $\mathcal{T}rop(I)$ is equivalent to the information of the supports of all polynomials in I, and also of the (standard) matroids of the vector spaces I_d^h. Theorem 1.1 is mostly proved in Section 2, though we postpone the discussion of valuated matroids, including recalling their definition, to Section 4. The version proved there (Theorem 4.2) also holds for a subscheme $Z \subset \mathbb{P}^n$ given by a homogeneous ideal in $K[x_0, \dots, x_n]$.

In Section 3 we show that the tropical cycle structure on a tropical variety $\operatorname{trop}(Y)$ can be recovered from its tropical scheme structure, answering the question raised in [GG16, Remark 7.2.3].

Theorem 1.2. *Let $Y \subset T$ be a subscheme defined by an ideal $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. The multiplicities of the maximal cells in the tropical variety $\operatorname{trop}(Y)$ can be recovered from the congruence $\mathcal{T}rop(I)$.*

The classical Hilbert-Chow morphism takes a subscheme of $\mathbb{P}^n$ to the associated cycle in the Chow group of $\mathbb{P}^n$. Theorem 1.2 can thus be thought of as a tropical version of this morphism.

Finally, in Section 4 we investigate in more depth the structure of the congruence $\mathcal{T}rop(I)$, and use ideas from valuated matroids and tropical linear spaces to characterize when a relation lives in $\mathcal{T}rop(I)$. We also show that any tropical polynomial has a distinguished representative in its equivalence class in $\mathcal{T}rop(I)$, and give a combinatorial procedure to compute it.

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2. Tropical varieties and their scheme structure

In this section we recall the necessary background on tropical geometry and the definition of the tropical scheme structure proposed in [GG16]. We also develop some fundamental properties of these congruences, leading to part of the proof of Theorem 1.1.

Throughout this paper we denote by $\overline{\mathbb{R}}$ the tropical semiring (or min-plus algebra)

$$\overline{\mathbb{R}} := (\mathbb{R} \cup \{\infty\}, \oplus, \odot),$$

where the tropical addition $\oplus$ is the usual minimum and tropical multiplication $\odot$ is the usual addition. We denote by $\mathbb{B}$ the Boolean subsemiring of $\overline{\mathbb{R}}$ consisting of $\{0, \infty\}$ with the induced operations. We denote by

$$S := \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \quad \text{and} \quad \tilde{S} := \overline{\mathbb{R}}[x_0, \dots, x_n]$$

the semirings of tropical Laurent polynomials and tropical polynomials in the variables $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{x} = (x_0, \dots, x_n)$, respectively. Elements of S or $\tilde{S}$ are (Laurent) polynomials with coefficients in $\overline{\mathbb{R}}$ where all operations are to be interpreted tropically. Explicitly, if $F \in \tilde{S}$ then F has the form $F(\mathbf{x}) = \bigoplus_{\mathbf{u} \in \mathbb{N}^{n+1}} a_{\mathbf{u}} \odot \mathbf{x}^{\mathbf{u}}$, where $a_{\mathbf{u}} \in \overline{\mathbb{R}}$ and all but finitely many of the $a_{\mathbf{u}}$ equal ∞. As a function, this is $F(\mathbf{x}) = \min_{\mathbf{u}\in\mathbb{N}^{n+1}}(a_{\mathbf{u}} + \mathbf{x}\cdot\mathbf{u})$. Elements of S have the form $F(\mathbf{x}) = \bigoplus_{\mathbf{u}\in\mathbb{Z}^n} a_{\mathbf{u}} \odot \mathbf{x}^{\mathbf{u}}$, where again $a_{\mathbf{u}} \in \overline{\mathbb{R}}$ and all but finitely many $a_{\mathbf{u}}$ equal ∞. Note that elements of S and $\tilde{S}$ are regarded as tropical polynomials, not functions. By this we mean that $F(x) = x^2 \oplus 0$ and $G(x) = x^2 \oplus 1 \odot x \oplus 0$ are different as elements of S, even though $F(w) = \min(2w, 0) = \min(2w, w+1, 0) = G(w)$ for all $w \in \mathbb{R}$.

We adopt the notational convention that lower case letters denote elements of the conventional (Laurent) polynomial ring with coefficients in K and upper case letters denote tropical (Laurent) polynomials with coefficients in $\overline{\mathbb{R}}$.

The *support* of a (Laurent) polynomial $f = \sum c_{\mathbf{u}}\mathbf{x}^{\mathbf{u}}$ is the subset of $\mathbb{N}^{n+1}$ (respectively $\mathbb{Z}^n$) defined by $\operatorname{supp}(f) := \{\mathbf{u} : c_{\mathbf{u}} \neq 0\}$. Similarly, for a tropical (Laurent) polynomial $F = \bigoplus a_{\mathbf{u}} \odot \mathbf{x}^{\mathbf{u}}$ we write $\operatorname{supp}(F) := \{\mathbf{u} : a_{\mathbf{u}} \neq \infty\}$. We call $a_{\mathbf{u}}$ the coefficient in F of the monomial $\mathbf{u}$.

Fix a field K with a valuation $\operatorname{val}\colon K \to \overline{\mathbb{R}}$. We write R for the valuation ring $\{c \in K : \operatorname{val}(c) \geq 0\}$, and $\Bbbk$ for the residue field $R/\{c \in K : \operatorname{val}(c) > 0\}$.

A polynomial $f \in K[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ gives rise to a tropical polynomial $\operatorname{trop}(f) \in S$ as follows. If $f = \sum_{\mathbf{u}\in\mathbb{Z}^n} c_{\mathbf{u}}\mathbf{x}^{\mathbf{u}}$, then

$$\operatorname{trop}(f) := \bigoplus_{\mathbf{u}\in\mathbb{Z}^n} \operatorname{val}(c_{\mathbf{u}}) \odot \mathbf{x}^{\mathbf{u}}.$$

The *tropical hypersurface* defined by f is

$$\operatorname{trop}(V(f)) := \{\mathbf{w} \in \mathbb{R}^n : \text{the minimum in } \operatorname{trop}(f)(\mathbf{w}) \text{ is achieved at least twice}\}.$$

The tropicalization of a variety $Y \subset (K^*)^n$ defined by an ideal $I \subset K[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ is

$$\operatorname{trop}(Y) := \bigcap_{f\in I} \operatorname{trop}(V(f)).$$

For more details on tropical varieties see [MS15].

Classically, a subscheme of the n-dimensional torus T is defined by an ideal in the Laurent polynomial ring $K[x_1^{\pm1}, \ldots, x_n^{\pm1}]$. There are two possible ways to tropicalize this. The first gives an ideal in the semiring S of tropical Laurent polynomials.

Definition 2.1. Let $I \subset K[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ be an ideal. The ideal $\operatorname{trop}(I)$ in the semiring S is generated by the tropical polynomials $\operatorname{trop}(f)$ for $f \in I$:

$$\operatorname{trop}(I) := \langle \operatorname{trop}(f) : f \in I \rangle.$$

The definition of $\operatorname{trop}(I)$ is the same for an ideal in $K[x_0, \ldots, x_n]$.

If the value group $\Gamma := \operatorname{im}\operatorname{val}$ of K equals all of $\overline{\mathbb{R}}$ and the residue field $\Bbbk$ is infinite then every tropical polynomial in the ideal $\operatorname{trop}(I)$ has the form $\operatorname{trop}(f)$ for some $f \in I$. Indeed, in that case $a \odot \mathbf{x}^{\mathbf{u}} \odot \operatorname{trop}(f) = \operatorname{trop}(c\,\mathbf{x}^{\mathbf{u}} f)$ for any $c \in K$ with $\operatorname{val}(c) = a$, and $\operatorname{trop}(f) \oplus \operatorname{trop}(g) = \operatorname{trop}(f + \alpha g)$ for a sufficiently general $\alpha \in K$ with $\operatorname{val}(\alpha) = 0$.

A different approach to tropicalizing the scheme defined by I is given in [GG16]. Here the ideal I gives rise to a *congruence* on S. This is an equivalence relation on S that is closed under tropical addition and tropical multiplication, i.e., if $F_1 \sim G_1$ and $F_2 \sim G_2$ then $F_1 \oplus F_2 \sim G_1 \oplus G_2$ and $F_1 \odot F_2 \sim G_1 \odot G_2$. If $\phi\colon S \to R$ is a semiring homomorphism, then $\{F \sim G : \phi(F) = \phi(G)\}$ is a congruence, and all congruences on S arise in this fashion. This is a key reason to consider congruences instead of only ideals. For a subset $\{(F_\alpha, G_\alpha) : \alpha \in A\}$ of $S \times S$ there is a smallest congruence on S containing $F_\alpha \sim G_\alpha$ for all $\alpha \in A$, which we denote by $\langle F_\alpha \sim G_\alpha \rangle_{\alpha \in A}$. All these notions also make sense for the semiring $\tilde{S}$.

The following definitions are taken from Definitions 5.1.1 and 6.1.3 of [GG16].

Definition 2.2. Let F be a tropical (Laurent) polynomial. For $\mathbf{v} \in \operatorname{supp}(F)$ we write $F_{\hat{\mathbf{v}}}$ for the tropical polynomial obtained by removing the term involving $\mathbf{v}$ from F. Explicitly, if $F = \bigoplus a_{\mathbf{u}} \odot \mathbf{x}^{\mathbf{u}}$, then

$$F_{\hat{\mathbf{v}}} := \bigoplus_{\mathbf{u} \neq \mathbf{v}} a_{\mathbf{u}} \odot \mathbf{x}^{\mathbf{u}}.$$

The *bend relations* of F are:

$$B(F) := \{F \sim F_{\hat{\mathbf{v}}} : \mathbf{v} \in \operatorname{supp}(F)\}.$$

Given an ideal $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the *scheme-theoretic tropicalization* of I is the congruence on S

$$\mathcal{T}rop(I) := \langle B(\operatorname{trop}(f)) : f \in I \rangle.$$

We use the same definition for $\mathcal{T}rop(I)$ if I is a homogeneous ideal in $K[x_0, \dots, x_n]$.

In [GG16] the authors show that the tropical variety of an ideal I can be recovered from the congruence $\mathcal{T}rop(I)$ as

$$\operatorname{trop}(V(I)) = \operatorname{Hom}(S/\mathcal{T}rop(I), \mathbb{R}),$$

where the homomorphisms are semiring homomorphisms. Explicitly, this means that

$$\operatorname{trop}(V(I)) = \{\mathbf{w} \in \mathbb{R}^n : \operatorname{trop}(f)(\mathbf{w}) = \operatorname{trop}(f)_{\hat{\mathbf{v}}}(\mathbf{w}) \text{ for all } f \in I, \mathbf{v} \in \operatorname{supp}(f)\}. \tag{2.1}$$

Remark 2.3. When I is a binomial ideal, an equivalent congruence appears in the work of Kahle and Miller [KM14]. For a binomial ideal $I \subset K[x_1, \dots, x_n]$ they define a congruence on the monoid $\mathbb{N}^n$ generated by the relations $\{\mathbf{u} \sim \mathbf{v} : \exists \lambda \in K^* \text{ such that } \mathbf{x}^{\mathbf{u}} - \lambda \mathbf{x}^{\mathbf{v}} \in I\}$. These relations, together with $\mathbf{u} \sim \infty$ whenever $\mathbf{x}^{\mathbf{u}} \in I$, generate the congruence $\mathcal{T}rop(I)$ on $\mathbb{B}[x_1, \dots, x_n]$ when K has the trivial valuation.

In the rest of this section we develop some basic properties of these congruences, leading to a proof of part of Theorem 1.1. We will make repeated use of the following result on congruences on S. By a monomial in S we mean a tropical polynomial whose support has size one.

Lemma 2.4. *The congruence $\langle F_\alpha \sim G_\alpha \rangle_{\alpha \in A}$ on S is equal to the transitive closure of the set U of relations of the form*

$$M \odot F_\alpha \oplus H \sim M \odot G_\alpha \oplus H$$

and their reverse, where $\alpha \in A$, $H \in S$, and M is a monomial in S.

Thus the congruence $\mathcal{T}rop(I)$ is equal to the transitive closure of the set of relations of the form

$$a \odot \operatorname{trop}(f)_{\hat{\mathbf{v}}} \oplus H \sim a \odot \operatorname{trop}(f) \oplus H$$

and their reverse, where $f \in I$, $\mathbf{v} \in \operatorname{supp}(f)$, $a \in \mathbb{R}$, *and* $H \in S$.

Proof. By [GG16, Lemma 2.4.5] we know that $\langle F_\alpha \sim G_\alpha \rangle_{\alpha \in A}$ is the transitive closure of the subsemiring of $S \times S$ generated by the elements $F_\alpha \sim G_\alpha$, $G_\alpha \sim F_\alpha$, and $1 \sim 1$. We first show that this is in fact the transitive closure T of the S-subsemimodule N (as opposed to the S-subsemiring) of $S \times S$ generated by these elements. Let $F \sim G$ and $F' \sim G'$ be elements of T. We will show that their tropical product and sum is also in T, and thus T is a subsemiring of $S \times S$, as desired. By definition, there exist chains $F \sim H_1 \sim \cdots \sim H_l \sim G$ and $F' \sim H'_1 \sim \cdots \sim H'_{l'} \sim G'$ of relations in N. The fact that $F \odot F' \sim G \odot G'$ is in T follows from the chain of relations in N

$$F \odot F' \sim H_1 \odot F' \sim \cdots \sim H_l \odot F' \sim G \odot F' \sim G \odot H'_1 \sim \cdots \sim G \odot H'_{l'} \sim G \odot G'.$$

A similar argument shows that the tropical sum $F \oplus F' \sim G \oplus G'$ is in T.

We now prove that all relations in N are in the transitive closure of the set U. Any relation in N has the form

$$\left(\bigoplus_{i=1}^{s} Q_i \odot C_i\right) \oplus Q \sim \left(\bigoplus_{i=1}^{s} Q_i \odot D_i\right) \oplus Q, \tag{2.2}$$

where all the Q_i are in S, all the relations $C_i \sim D_i$ have the form $F_\alpha \sim G_\alpha$ or their reverse, and $Q \in S$. By allowing some of the relations $C_i \sim D_i$ to be equal, we can assume that the Q_i are monomials in S. For $l = 0, 1, \ldots, s$, let $H_l \in S$ be defined by

$$H_l := \left(\bigoplus_{i=1}^{l} Q_i \odot D_i\right) \oplus \left(\bigoplus_{i=l+1}^{s} Q_i \odot C_i\right) \oplus Q.$$

Note that $H_0 \sim H_1 \sim \cdots \sim H_s$ is a chain of relations in U. The relation (2.2) is simply $H_0 \sim H_s$, so it is in the transitive closure of U.

The last claim of the lemma follows from the fact that $\mathcal{T}rop(I)$ is generated by the relations $\operatorname{trop}(f) \sim \operatorname{trop}(f)_{\hat{\mathbf{v}}}$ for $f \in I$. If $f \in I$ then $\mathbf{x}^{\mathbf{u}} f \in I$, so we may replace the tropical monomial M by a scalar a. ☐

Remark 2.5. If the value group $\Gamma = \operatorname{im}\operatorname{val}$ equals all of $\overline{\mathbb{R}}$ then for all scalars $a \in \mathbb{R}$ we can find $c \in K$ with $\operatorname{val}(c) = a$, so $a \odot \operatorname{trop}(f) = \operatorname{trop}(cf)$. Therefore, in this case the congruence $\mathcal{T}rop(I)$ can be described as the transitive closure of the set of relations of the form $\operatorname{trop}(f)_{\hat{\mathbf{v}}} \oplus H \sim \operatorname{trop}(f) \oplus H$ and their reverse, where $f \in I$, $\mathbf{v} \in \operatorname{supp}(f)$, and $H \in S$.

The following proposition is the key technical result that is needed to prove Theorem 1.2 and parts of Theorem 1.1.

Proposition 2.6. *Let* I *be an ideal in* $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, *and let* $F \sim G$ *be a relation in the congruence* $\mathcal{T}rop(I)$ *on* S, *where* $F = \bigoplus \alpha_{\mathbf{u}} \odot \mathbf{x}^{\mathbf{u}}$ *and* $G = \bigoplus \beta_{\mathbf{u}} \odot \mathbf{x}^{\mathbf{u}}$. *Then there is a chain* $F = F_0 \sim F_1 \sim \cdots \sim F_s \sim F_{s+1} = G$ *of relations in* $\mathcal{T}rop(I)$ *satisfying the following two properties.*

(a) *Each* $F_i \sim F_{i+1}$ *has the form* $m \odot \operatorname{trop}(g) \oplus H \sim m \odot \operatorname{trop}(g)_{\hat{\mathbf{v}}} \oplus H$ *or the reverse, for some* $g \in I$, $\mathbf{v} \in \operatorname{supp}(g)$, $H \in S$, *and* $m \in \mathbb{R}$.

(b) *The coefficient* $\gamma_{\mathbf{u},i}$ *of* $\mathbf{u}$ *in* F_i *equals either* $\alpha_{\mathbf{u}}$ *or* $\beta_{\mathbf{u}}$.

Proof. By Lemma 2.4 there is a chain $F = F_0 \sim F_1 \sim \cdots \sim F_s \sim F_{s+1} = G$ of relations in $\mathcal{T}rop(I)$ with the property that for each i we have $F_i \sim F_{i+1}$ equal to $m_i \odot \operatorname{trop}(g_i) \oplus H_i \sim m_i \odot \operatorname{trop}(g_i)_{\hat{\mathbf{v}}} \oplus H_i$ or the reverse, for some polynomial $g_i \in I$,

$\mathbf{v} \in \operatorname{supp}(g_i)$, $H_i \in S$, and $m_i \in \mathbb{R}$. We now show that we can modify this chain to get a chain where the coefficients have the required form. We represent the given chain by a path of length $s+1$ with vertices labeled by the F_i and an oriented edge labeled by $\mathbf{v}$ from $m_i \odot \operatorname{trop}(g_i) \oplus H_i$ to $m_i \odot \operatorname{trop}(g_i)_{\hat{\mathbf{v}}} \oplus H_i$.

We claim that we can locally modify the path by switching the order of adjacent edges or amalgamating edges if the labels agree, in the following six ways:

(1) $F_{i-1} \xrightarrow{\mathbf{v}} F_i \xleftarrow{\mathbf{v}'} F_{i+1}$ can be replaced by $F_{i-1} \xleftarrow{\mathbf{v}'} F'_i \xrightarrow{\mathbf{v}} F_{i+1}$,
(2) $F_{i-1} \xrightarrow{\mathbf{v}} F_i \xrightarrow{\mathbf{v}'} F_{i+1}$ can be replaced by $F_{i-1} \xrightarrow{\mathbf{v}'} F'_i \xrightarrow{\mathbf{v}} F_{i+1}$,
(3) $F_{i-1} \xleftarrow{\mathbf{v}} F_i \xleftarrow{\mathbf{v}'} F_{i+1}$ can be replaced by $F_{i-1} \xleftarrow{\mathbf{v}'} F'_{i+1} \xleftarrow{\mathbf{v}} F_{i+1}$,
(4) $F_{i-1} \xleftarrow{\mathbf{v}} F_i \xrightarrow{\mathbf{v}} F_{i+1}$ can be replaced by one of $F_{i-1} \xleftarrow{\mathbf{v}} F_{i+1}$, $F_{i-1} \xrightarrow{\mathbf{v}} F_{i+1}$, or $F_{i-1} = F_{i+1}$,
(5) $F_{i-1} \xrightarrow{\mathbf{v}} F_i \xrightarrow{\mathbf{v}} F_{i+1}$ can be replaced by $F_{i-1} \xrightarrow{\mathbf{v}} F_{i+1}$, and
(6) $F_{i-1} \xleftarrow{\mathbf{v}} F_i \xleftarrow{\mathbf{v}} F_{i+1}$ can be replaced by $F_{i-1} \xleftarrow{\mathbf{v}} F_{i+1}$.

By repeated use of the first of these operations we may assume that all left-pointing arrows in the path come before all right-pointing arrows. If $\mathbf{v}$ appears as an arrow label on more than one arrow, by repeated use of the second and third operations we may assume that the left-pointing arrows labeled by $\mathbf{v}$ are the last left-pointing arrows, and the right-pointing arrows labeled by $\mathbf{v}$ are the first right-pointing arrows. By repeated use of the last three operations we can then replace these arrows by at most one arrow labeled by $\mathbf{v}$. In this fashion we get a new chain $F = F_0 \sim F_1 \sim \cdots \sim F_s \sim F_{s+1} = G$ where each arrow label occurs exactly once. As the coefficient of $\mathbf{v}$ in F_i equals that in F_{i+1} unless the arrow between F_i and F_{i+1} is labeled by $\mathbf{v}$, this means that the coefficient of $\mathbf{v}$ changes at most once in the path from F to G, so for all F_i the coefficient of $\mathbf{v}$ equals the coefficient of $\mathbf{v}$ in either F or G.

It thus suffices to show that these six arrow replacements can be made. In each case we make use of the fact that the coefficients of F_{i-1}, F_i and F_{i+1} all agree for monomials $\mathbf{u} \neq \mathbf{v}, \mathbf{v}'$.

• **Case** $F_{i-1} \xrightarrow{\mathbf{v}} F_i \xleftarrow{\mathbf{v}'} F_{i+1}$. By assumption we have $F_{i-1} = m_{i-1} \odot \operatorname{trop}(g_{i-1}) \oplus H_{i-1}$, $F_i = m_{i-1} \odot \operatorname{trop}(g_{i-1})_{\hat{\mathbf{v}}} \oplus H_{i-1} = m_{i+1} \odot \operatorname{trop}(g_{i+1})_{\hat{\mathbf{v}}'} \oplus H_{i+1}$, and $F_{i+1} = m_{i+1} \odot \operatorname{trop}(g_{i+1}) \oplus H_{i+1}$. Let $c_{\mathbf{v}}$ be the coefficient of $\mathbf{x}^{\mathbf{v}}$ in g_{i-1}, and let $d_{\mathbf{v}'}$ be the coefficient of $\mathbf{x}^{\mathbf{v}'}$ in g_{i+1}. Let $H'_{i-1} = H_{i-1} \oplus m_{i+1} \odot \operatorname{val}(d_{\mathbf{v}'}) \odot \mathbf{x}^{\mathbf{v}'}$, and $H'_{i+1} = H_{i+1} \oplus m_{i-1} \odot \operatorname{val}(c_{\mathbf{v}}) \odot \mathbf{x}^{\mathbf{v}}$. Set $F'_i = m_{i-1} \odot \operatorname{trop}(g_{i-1}) \oplus H'_{i-1}$, and note that this equals $m_{i+1} \odot \operatorname{trop}(g_{i+1}) \oplus H'_{i+1}$. In addition, $F_{i-1} = m_{i+1} \odot \operatorname{trop}(g_{i+1})_{\hat{\mathbf{v}}'} \oplus H'_{i+1}$ and $F_{i+1} = m_{i-1} \odot \operatorname{trop}(g_{i-1})_{\hat{\mathbf{v}}} \oplus H'_{i-1}$. We then have the relationship $F_{i-1} \xleftarrow{\mathbf{v}'} F'_i \xrightarrow{\mathbf{v}} F_{i+1}$ as required.

• **Case** $F_{i-1} \xrightarrow{\mathbf{v}} F_i \xrightarrow{\mathbf{v}'} F_{i+1}$. By assumption $F_{i-1} = m_{i-1} \odot \operatorname{trop}(g_{i-1}) \oplus H_{i-1}$, $F_i = m_{i-1} \odot \operatorname{trop}(g_{i-1})_{\hat{\mathbf{v}}} \oplus H_{i-1} = m_i \odot \operatorname{trop}(g_i) \oplus H_i$, and $F_{i+1} = m_i \odot \operatorname{trop}(g_i)_{\hat{\mathbf{v}}'} \oplus H_i$. Write $g_{i-1} = \sum c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}$, and $g_i = \sum d_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}$.

Set $H'_i = H_i \oplus m_{i-1} \odot \operatorname{val}(c_{\mathbf{v}}) \odot \mathbf{x}^{\mathbf{v}}$. We have $F_{i-1} = m_i \odot \operatorname{trop}(g_i) \oplus H'_i$. Set $F'_i = m_i \odot \operatorname{trop}(g_i)_{\hat{\mathbf{v}}'} \oplus H'_i$. We now have two further subcases. Let $b_{\mathbf{v}'}$ be the coefficient of $\mathbf{v}'$ in H_i.

(1) $b_{\mathbf{v}'} \leq m_{i-1} + \mathrm{val}(c_{\mathbf{v}'})$. Set $H'_{i-1} = (H_{i-1})_{\hat{\mathbf{v}}'} \oplus b_{\mathbf{v}'} \odot \mathbf{x}^{\mathbf{v}'}$. In this case F'_i equals $m_{i-1} \odot \mathrm{trop}(g_{i-1}) \oplus H'_{i-1}$. We then have $F_{i+1} = m_{i-1} \odot \mathrm{trop}(g_{i-1})_{\hat{\mathbf{v}}} \oplus H'_{i-1}$, and thus $F_{i-1} \xrightarrow{\mathbf{v}'} F'_i \xrightarrow{\mathbf{v}} F_{i+1}$ as required.

(2) $b_{\mathbf{v}'} > m_{i-1} + \mathrm{val}(c_{\mathbf{v}'})$. This implies in particular that $c_{\mathbf{v}'} \neq 0$. Let $h = g_{i-1} - (c_{\mathbf{v}'}/d_{\mathbf{v}'})g_i = \sum (c_{\mathbf{u}} - d_{\mathbf{u}}(c_{\mathbf{v}'}/d_{\mathbf{v}'}))\mathbf{x}^{\mathbf{u}}$. By construction $\mathbf{v}' \notin \mathrm{supp}(h)$. Set $H'_{i+1} = F_{i+1}$.

We claim that $F'_i = m_{i-1} \odot \mathrm{trop}(h) \oplus H'_{i+1}$. The coefficient of $\mathbf{v}'$ in F_i is $m_i + \mathrm{val}(d_{\mathbf{v}'})$, since $F_i \neq F_{i+1}$, so comparing the two different expressions for F_i we see that $m_i + \mathrm{val}(d_{\mathbf{v}'}) \leq m_{i-1} + \mathrm{val}(c_{\mathbf{v}'})$. Thus $\mathrm{val}(c_{\mathbf{v}'}/d_{\mathbf{v}'}) \geq m_i - m_{i-1}$. The coefficient of $\mathbf{u}$ in $m_{i-1} \odot \mathrm{trop}(h)$ is $m_{i-1} + \mathrm{val}(c_{\mathbf{u}} - d_{\mathbf{u}}(c_{\mathbf{v}'}/d_{\mathbf{v}'}))$, which is at least $m_{i-1} + \min(\mathrm{val}(c_{\mathbf{u}}), \mathrm{val}(d_{\mathbf{u}}) + \mathrm{val}(c_{\mathbf{v}'}/d_{\mathbf{v}'}))$. This in turn is at least $\min(m_{i-1} + \mathrm{val}(c_{\mathbf{u}}), m_i + \mathrm{val}(d_{\mathbf{u}}))$. For $\mathbf{u} \neq \mathbf{v}, \mathbf{v}'$ both terms in this minimum are at least the coefficient of $\mathbf{u}$ in F_{i-1}, which equals that in $F_{i+1} = H'_{i+1}$. For $\mathbf{u} = \mathbf{v}$, $m_{i-1} + \mathrm{val}(c_{\mathbf{v}}) < m_i + \mathrm{val}(d_{\mathbf{v}})$, since $F_{i-1} \neq F_i$, so $\mathrm{val}(c_{\mathbf{v}} - d_{\mathbf{v}}(c_{\mathbf{v}'}/d_{\mathbf{v}'})) = \mathrm{val}(c_{\mathbf{v}})$. The coefficient of $\mathbf{v}$ in $m_{i-1} \odot \mathrm{trop}(h) \oplus H'_{i+1}$ is then equal to $m_{i-1} + \mathrm{val}(c_{\mathbf{v}})$, which equals the coefficient in F'_i. Finally, the coefficient of $\mathbf{v}'$ is $b_{\mathbf{v}'}$, which is also the coefficient of $\mathbf{v}'$ in F'_i.

Since the coefficient of $\mathbf{v}$ is $m_{i-1} + \mathrm{val}(c_{\mathbf{v}}) < \infty$, we have $\mathbf{v} \in \mathrm{supp}(\mathrm{trop}(h))$, and thus $F_{i+1} = m_{i-1} \odot \mathrm{trop}(h)_{\hat{\mathbf{v}}} \oplus H'_{i+1}$. This again gives the relation $F_{i-1} \xrightarrow{\mathbf{v}'} F'_i \xrightarrow{\mathbf{v}} F_{i+1}$.

- **Case** $F_{i-1} \xleftarrow{\mathbf{v}} F_i \xleftarrow{\mathbf{v}'} F_{i+1}$. This is identical to the previous case with the roles of F_{i-1} and F_{i+1} reversed.

- **Case** $F_{i-1} \xleftarrow{\mathbf{v}} F_i \xrightarrow{\mathbf{v}} F_{i+1}$. By assumption we have $F_i = m_i \odot \mathrm{trop}(g_i) \oplus H_i = m'_i \odot \mathrm{trop}(g'_i) \oplus H'_i$, $F_{i-1} = m_i \odot \mathrm{trop}(g_i)_{\hat{\mathbf{v}}} \oplus H_i$, and $F_{i+1} = m'_i \odot \mathrm{trop}(g_{i+1})_{\hat{\mathbf{v}}} \oplus H'_i$. Let $\gamma_{\mathbf{v},j}$ be the coefficient of $\mathbf{v}$ in F_j, for $j = i-1, i, i+1$. We have $\gamma_{\mathbf{v},i-1}, \gamma_{\mathbf{v},i+1} > \gamma_{\mathbf{v},i}$. If $\gamma_{\mathbf{v},i-1} = \gamma_{\mathbf{v},i+1}$ then $F_{i-1} = F_{i+1}$, so we may replace $F_{i-1} \xleftarrow{\mathbf{v}} F_i \xrightarrow{\mathbf{v}} F_{i+1}$ by just $F_{i-1} = F_{i+1}$. If $\gamma_{\mathbf{v},i-1} < \gamma_{\mathbf{v},i+1}$, let $\tilde{H}_i = F_{i+1}$. Then $F_{i-1} = (\gamma_{\mathbf{v},i-1} - \gamma_{\mathbf{v},i}) \odot m'_i \odot \mathrm{trop}(g'_i) \oplus \tilde{H}_i$, and $F_{i+1} = (\gamma_{\mathbf{v},i-1} - \gamma_{\mathbf{v},i}) \odot m'_i \odot \mathrm{trop}(g'_i)_{\hat{\mathbf{v}}} \oplus \tilde{H}_i$, so we can replace $F_{i-1} \xleftarrow{\mathbf{v}} F_i \xrightarrow{\mathbf{v}} F_{i+1}$ by $F_{i-1} \xrightarrow{\mathbf{v}} F_{i+1}$. If $\gamma_{\mathbf{v},i-1} > \gamma_{\mathbf{v},i+1}$ then with the same construction we can replace $F_{i-1} \xleftarrow{\mathbf{v}} F_i \xrightarrow{\mathbf{v}} F_{i+1}$ by $F_{i-1} \xleftarrow{\mathbf{v}} F_{i+1}$.

- **Case** $F_{i-1} \xrightarrow{\mathbf{v}} F_i \xrightarrow{\mathbf{v}} F_{i+1}$. By assumption $F_{i-1} = m_{i-1} \odot \mathrm{trop}(g_{i-1}) \oplus H_{i-1}$, $F_i = m_{i-1} \odot \mathrm{trop}(g_{i-1})_{\hat{\mathbf{v}}} \oplus H_{i-1} = m_i \odot \mathrm{trop}(g_i) \oplus H_i$, and $F_{i+1} = m_i \odot \mathrm{trop}(g_i)_{\hat{\mathbf{v}}} \oplus H_i$. Set $H'_i = F_{i+1}$. Then $F_{i-1} = m_{i-1} \odot \mathrm{trop}(g_{i-1}) \oplus H'_i$, and $F_{i+1} = m_{i-1} \odot \mathrm{trop}(g_{i-1})_{\hat{\mathbf{v}}} \oplus H'_i$, so we may replace $F_{i-1} \xrightarrow{\mathbf{v}} F_i \xrightarrow{\mathbf{v}} F_{i+1}$ by $F_{i-1} \xrightarrow{\mathbf{v}} F_{i+1}$.

- **Case** $F_{i-1} \xleftarrow{\mathbf{v}} F_i \xleftarrow{\mathbf{v}} F_{i+1}$. This is identical to the previous case, with the roles of F_{i-1} and F_{i+1} reversed. □

The first of the six cases in the previous proof is the only one that cannot be reversed. Indeed, in the congruence $\mathcal{T}rop(\langle x+y \rangle)$ we have the relations $x \sim x \oplus y \sim y$ but neither $x \sim \infty$ nor $y \sim \infty$.

A congruence J on $\tilde{S}$ or S is *homogeneous* with respect to a grading by $\deg(x_i) = \delta_i \in \mathbb{Z}$ if J is generated by relations of the form $F \sim G$ where F and G are both homogeneous of the same degree. For a tropical polynomial F we write F_d for its homogeneous component of degree d, where $d \in \mathbb{Z}$.

Proposition 2.7. *Let J be a homogeneous congruence on $\tilde{S}$ or S. If $F \sim G \in J$, then $F_d \sim G_d \in J$ for all $d \in \mathbb{Z}$.*

Proof. Let $\mathcal{J} = \{F_\alpha \sim G_\alpha : \alpha \in A\}$ be a homogeneous generating set for J, and fix $F \sim G \in J$. By Lemma 2.4 there is a chain $F = F_0 \sim F_1 \sim \cdots \sim F_s \sim F_{s+1} = G$ of relations in J where $F_i \sim F_{i+1}$ has the form $M_i \odot H_i \oplus P_i \sim M_i \odot H_i' \oplus P_i$ with $H_i \sim H_i' \in \mathcal{J}$ (or its reverse), M_i a monomial in S, and $P_i \in S$. The dth graded piece of $F_i \sim F_{i+1}$ either has the form $Q_i \sim Q_i$ or $M_i \odot H_i \oplus Q_i \sim M_i \odot H_i' \oplus Q_i$ where Q_i is homogeneous of degree d. Thus $(F_i)_d \sim (F_{i+1})_d \in J$, so $F_d \sim G_d \in J$. ☐

We will also need the notion of the homogenization of a congruence on S. This will play the same role as the homogenization of an ideal in the usual Laurent polynomial ring with coefficients in K. Geometrically, this is the tropical analogue of taking the projective closure of a subvariety of $(K^*)^n$.

Recall that the homogenization of a polynomial $f = \sum c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in K[x_1, \ldots, x_n]$ is the polynomial $\tilde{f} := \sum c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} x_0^{\deg(f)-|\mathbf{u}|} \in K[x_0, \ldots, x_n]$, where $|\mathbf{u}| = u_1 + \cdots + u_n$, and $\deg(f)$ is the maximum of $|\mathbf{u}|$ for which $c_{\mathbf{u}} \neq 0$. For an ideal $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, its homogenization is the ideal $I^h = \langle \tilde{f} : f \in I \cap K[x_1, \ldots, x_n] \rangle$.

Definition 2.8. Given $F = \bigoplus a_{\mathbf{u}} \odot \mathbf{x}^{\mathbf{u}} \in S$, we denote by $\deg(F)$ the maximum $\max(|\mathbf{u}| : a_{\mathbf{u}} \neq \infty)$. If $\mathrm{supp}(F) \subset \mathbb{N}^n$, we write $\tilde{F}$ for the tropical polynomial in $\tilde{S}$ given by

$$\tilde{F} := \bigoplus a_{\mathbf{u}} \odot \mathbf{x}^{\mathbf{u}} \odot x_0^{\deg(F)-|\mathbf{u}|}.$$

If $F \sim G$ is a relation, where F, G are tropical polynomials (with $\mathrm{supp}(F), \mathrm{supp}(G) \subset \mathbb{N}^n$) satisfying $\deg(F) \geq \deg(G)$, its homogenization is

$$\widetilde{F \sim G} \quad := \quad \tilde{F} \sim x_0^{\deg(F)-\deg(G)} \odot \tilde{G}.$$

Let J be a congruence on S. The homogenization J^h of J is the congruence

$$J^h := \langle \widetilde{F \sim G} : F \sim G \in J \text{ and } \mathrm{supp}(F), \mathrm{supp}(G) \subset \mathbb{N}^n \rangle.$$

Proposition 2.9. *Let I be an ideal in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and let $I^h \subset K[x_0, \ldots, x_n]$ be its homogenization. Then we have the equality of congruences on $\tilde{S}$*

$$\mathcal{T}rop(I^h) = \mathcal{T}rop(I)^h.$$

Proof. Let f be a homogeneous polynomial in I^h. Write $g = f|_{x_0=1}$. Note that $g \in I$, and $f = x_0^a \tilde{g}$ for some $a \geq 0$. For a monomial $\mathbf{x}^{\mathbf{u}} \in K[x_0, \ldots, x_n]$ write $\mathbf{u}'$ for the projection of $\mathbf{u}$ onto the last n coordinates.

Choose $\mathbf{u} \in \mathrm{supp}(f)$, and consider the relation $\mathrm{trop}(f) \sim \mathrm{trop}(f)_{\hat{\mathbf{u}}} \in \mathcal{T}rop(I^h)$. The homogenization of the relation $\mathrm{trop}(g) \sim \mathrm{trop}(g)_{\hat{\mathbf{u}}'} \in \mathcal{T}rop(I)$ is

$$\widetilde{\mathrm{trop}(g)} \sim x_0^{\deg(g)-\deg(g_{\hat{\mathbf{u}}'})} \odot \widetilde{\mathrm{trop}(g)_{\hat{\mathbf{u}}'}}.$$

Multiplying this relation by x_0^a gives the relation $\mathrm{trop}(f) \sim \mathrm{trop}(f)_{\hat{\mathbf{u}}}$. Since these relations generate $\mathcal{T}rop(I^h)$, it follows that $\mathcal{T}rop(I^h) \subset \mathcal{T}rop(I)^h$.

For the converse, it suffices to consider a relation of the form $\widetilde{F \sim G}$ for $F \sim G \in \mathcal{T}rop(I)$ with both F, G non-Laurent tropical polynomials, and show that it is a relation in $\mathcal{T}rop(I^h)$. By Proposition 2.6 we can find a chain $F = F_0 \sim F_1 \sim \cdots \sim F_s \sim F_{s+1} = G$ with $F_i \sim F_{i+1} \in \mathcal{T}rop(I)$ of the form $a_i \odot \operatorname{trop}(h_i) \oplus H_i \sim a_i \odot \operatorname{trop}(h_i)_{\hat{\mathbf{v}}} \oplus H_i$ for some $h_i \in I$, $a_i \in \mathbb{R}$, and $H_i \in S$, and for which the coefficient of $\mathbf{u}$ in F_i equals the coefficient of $\mathbf{u}$ in either F or G. This latter condition implies that if $\mathbf{u} \notin \operatorname{supp}(F) \cup \operatorname{supp}(G)$ then $\mathbf{u} \notin \operatorname{supp}(F_i)$, so in particular each F_i has support in $\mathbb{N}^n$ and $\deg(F_i) \leq \max(\deg(F), \deg(G))$. The homogenization of the relation $a_i \odot \operatorname{trop}(h_i) \oplus H_i \sim a_i \odot \operatorname{trop}(h_i)_{\hat{\mathbf{v}}} \oplus H_i$ equals

$$a_i \odot \operatorname{trop}(\tilde{h}_i) \odot x_0^b \oplus \tilde{H}_i \odot x_0^d \sim a_i \odot \operatorname{trop}(\tilde{h}_i)_{\hat{\mathbf{v}}'} \odot x_0^b \oplus \tilde{H}_i \odot x_0^d,$$

where $\mathbf{v}' \in \mathbb{N}^{n+1}$ has last n coordinates equal to $\mathbf{v}$, and the numbers b, d satisfy $b = \max(\deg(H_i) - \deg(h_i), 0)$ and $d = \max(\deg(h_i) - \deg(H_i), 0)$. Since $\tilde{h}_i \in I^h$, we have $\operatorname{trop}(\tilde{h}_i) \sim \operatorname{trop}(\tilde{h}_i)_{\mathbf{v}'} \in \mathcal{T}rop(I^h)$, and so $\widetilde{F_i \sim F_{i+1}} \in \mathcal{T}rop(I^h)$.

Each relation $\widetilde{F_i \sim F_{i+1}}$ is homogeneous of degree at most $\max(\deg(F), \deg(G))$. The right-hand side of $\widetilde{F_{i-1} \sim F_i}$ and the left-hand side of $\widetilde{F_i \sim F_{i+1}}$ are either identical or differ by a factor of x_0^b, with $b \in \mathbb{N}$ equal to the difference between their degrees. Thus we can multiply both sides of the lower degree relation by x_0^b to get two relations whose adjacent terms coincide. Doing this for the string $\widetilde{F_0 \sim F_1}, \ldots, \widetilde{F_s \sim F_{s+1}}$ gives a chain of relations in $\mathcal{T}rop(I^h)$ of the same degree, whose first entry is $x_0^a \odot \widetilde{F}$ and whose last entry is $x_0^b \odot \widetilde{G}$ for some $a, b \in \mathbb{N}$ with at most one of a and b nonzero. Taking the transitive closure we get $x_0^a \odot \widetilde{F} \sim x_0^b \odot \widetilde{G} \in \mathcal{T}rop(I^h)$. This relation equals $\widetilde{F \sim G}$, which completes the proof. $\square$

Note that the use of Proposition 2.6 was key in the proof of Proposition 2.9.

We are now in position to prove the equivalence (1) $\Leftrightarrow$ (2) of Theorem 1.1 from the introduction.

Proof of (1) $\Leftrightarrow$ (2) of Theorem 1.1. We first show that the ideal $\operatorname{trop}(I)$ determines the congruence $\mathcal{T}rop(I)$. Specifically, we will show that

$$\mathcal{T}rop(I) = \langle F \sim F_{\hat{\mathbf{u}}} : F \in \operatorname{trop}(I), \mathbf{u} \in \operatorname{supp}(F) \rangle.$$

The inclusion $\subseteq$ follows from the fact that $\mathcal{T}rop(I)$ is generated by relations of the form $\operatorname{trop}(f) \sim \operatorname{trop}(f)_{\hat{\mathbf{u}}}$ for $f \in I$, which have the form $F \sim F_{\hat{\mathbf{u}}}$ for $F \in \operatorname{trop}(I)$. To prove the reverse inclusion, note that any $F \in \operatorname{trop}(I)$ has the form $\bigoplus_{1 \leq i \leq s} a_i \odot \operatorname{trop}(f_i)$ for some $f_1, \ldots, f_s \in I$ and $a_i \in \mathbb{R}$. If $\mathbf{u} \in \operatorname{supp}(F)$, the polynomial $F_{\hat{\mathbf{u}}}$ is $\bigoplus_{1 \leq i \leq s} a_i \odot \operatorname{trop}(f_i)_{\hat{\mathbf{u}}}$, where we set $\operatorname{trop}(f_i)_{\hat{\mathbf{u}}} = \operatorname{trop}(f_i)$ if $\mathbf{u} \notin \operatorname{supp}(f_i)$. Thus $F \sim F_{\hat{\mathbf{u}}}$ equals the tropical sum of $a_i \odot \operatorname{trop}(f_i) \sim a_i \odot \operatorname{trop}(f_i)_{\hat{\mathbf{u}}}$ for $1 \leq i \leq s$, and so it lies in $\mathcal{T}rop(I)$.

To show that the congruence $\mathcal{T}rop(I)$ determines the ideal $\operatorname{trop}(I)$, first note that by Proposition 2.9 the congruence $\mathcal{T}rop(I)$ determines the congruence $\mathcal{T}rop(I^h)$ on $\tilde{S}$, where $I^h \subset K[x_0, \ldots, x_n]$ is the homogenization of the ideal I. For any $d \geq 0$, denote by $\operatorname{trop}(I^h)_d$ the degree d part of the homogeneous ideal $\operatorname{trop}(I^h) \subset \tilde{S}$. We may regard any homogeneous tropical polynomial F of degree d as a tropical linear form l_F on the space $\overline{\mathbb{R}}^{\binom{n+d}{d}}$, whose coordinates are indexed by the monomials of degree d. Under this identification the set $\operatorname{trop}(I^h)_d$ is a tropical linear space; in fact, $\operatorname{trop}(I^h)_d$ is equal to the tropicalization $\operatorname{trop}(I^h_d)$ of the degree-d part I^h_d of I^h. Let ℓ_d be the tropical linear space in $\overline{\mathbb{R}}^{\binom{n+d}{d}}$ on which the tropical linear forms l_F vanish for all $F \in \operatorname{trop}(I^h)_d$, i.e., ℓ_d is the set of $\mathbf{z} \in \overline{\mathbb{R}}^{\binom{n+d}{d}}$ where the minimum in $l_F(\mathbf{z})$ is achieved twice for all $F \in \operatorname{trop}(I^h)_d$. We have that ℓ_d is the dual tropical linear space $(\operatorname{trop}(I^h)_d)^\perp$. The collection of tropical linear forms l_G that vanish on ℓ_d is $\ell_d^\perp = (\operatorname{trop}(I^h)_d)^{\perp\perp}$, which is equal to $\operatorname{trop}(I^h)_d$ [Rin12, Corollary 6.15]. Note that if G is a homogeneous tropical polynomial of degree d, the tropical linear form l_G vanishes on ℓ_d if and only if $l_G(\mathbf{z}) = l_{G_{\hat{\mathbf{u}}}}(\mathbf{z})$ for all $\mathbf{z} \in \ell_d$ and $\mathbf{u} \in \operatorname{supp}(G)$. It follows that if $G \in \tilde{S}_d$ is such that its bend relations $B(G)$ are in $\mathcal{T}rop(I^h)_d$, then $G \in \ell_d^\perp = \operatorname{trop}(I^h)_d$. We therefore have

$$\operatorname{trop}(I^h)_d = \{G \in \tilde{S}_d : B(G) \subset \mathcal{T}rop(I^h)_d\},$$

which shows that $\mathcal{T}rop(I^h)$ determines $\operatorname{trop}(I^h)_d$ for all $d \geq 0$, and thus $\operatorname{trop}(I^h)$. Since $\operatorname{trop}(I)$ is the ideal in S generated by $\{F|_{x_0=0} : F \in \operatorname{trop}(I^h)\}$, it also determines $\operatorname{trop}(I)$. $\square$

Remark 2.10. The tropical linear spaces ℓ_d also encode the valuated matroids of the vector spaces I^h_d, so we can see the third equivalence of Theorem 1.1 from the previous argument as well. This is elaborated on in Section 4.

3. Multiplicities

In this section we prove Theorem 1.2. The strategy is to define a Gröbner theory for congruences on the semiring of tropical polynomials, which lets us determine the multiplicities from the tropical scheme.

We first recall the definition of multiplicity for maximal cells of a tropical variety. For an irreducible d-dimensional subvariety $Y \subset (K^*)^n$ the tropical variety $\operatorname{trop}(Y) \subset \mathbb{R}^n$ is the support of a pure d-dimensional Γ-rational polyhedral complex. This means that $\operatorname{trop}(Y)$ is the union of a set Σ of d-dimensional polyhedra of the form $\{\mathbf{w} \in \mathbb{R}^n : A\mathbf{w} \leq \mathbf{b}\}$ where $A \in \mathbb{Q}^{r\times n}$ and $\mathbf{b} \in \Gamma^r$ for some $r \in \mathbb{N}$, and these polyhedra intersect only along faces. See [MS15, Chapter 3] for more details.

Let $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be the ideal of Y. Fix a group homomorphism $\Gamma \to K^*$, which we write $w \mapsto t^w$, satisfying $\operatorname{val}(t^w) = w$. This may require replacing K by an extension field; see [MS15, Chapter 2]. For a in the valuation ring R we write $\overline{a}$ for its image in the residue field $\Bbbk$. Fix $\mathbf{w}$ in the relative interior of a d-dimensional polyhedron $\sigma \in \Sigma$. We denote by $\operatorname{in}_{\mathbf{w}}(I) \subset \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ the initial ideal of I with respect to $\mathbf{w}$, in the sense described in [MS15, Section 2.4]. This is the ideal $\operatorname{in}_{\mathbf{w}}(I) := \langle \operatorname{in}_{\mathbf{w}}(f) : f \in I\rangle$, where for $f = \sum c_{\mathbf{u}}\mathbf{x}^{\mathbf{u}}$ the initial form $\operatorname{in}_{\mathbf{w}}(f)$ equals $\sum_{\operatorname{val}(c_{\mathbf{u}})+\mathbf{w}\cdot\mathbf{u}=\gamma} \overline{t^{-\operatorname{val}(c_{\mathbf{u}})}c_{\mathbf{u}}}\, \mathbf{x}^{\mathbf{u}}$, with $\gamma = \min(\operatorname{val}(c_{\mathbf{u}}) + \mathbf{w}\cdot\mathbf{u}) = \operatorname{trop}(f)(\mathbf{w})$.

The multiplicity of $\mathbf{w}$ is the multiplicity of the initial ideal $\operatorname{in}_{\mathbf{w}}(I)$:

$$\operatorname{mult}(\mathbf{w}) := \sum_P \operatorname{mult}(P, \operatorname{in}_{\mathbf{w}}(I)),$$

where the sum is over the minimal associated primes of $\mathrm{in}_{\mathbf{w}}(I)$, and $\mathrm{mult}(P, \mathrm{in}_{\mathbf{w}}(I))$ is the multiplicity of the associated primary component. See [MS15, Section 3.4] for more details. If coordinates on the torus $(K^*)^n$ have been chosen so that $\mathrm{in}_{\mathbf{w}}(I)$ has a generating set involving only the variables $x_{d+1}, \ldots, x_n$, then

$$\mathrm{mult}(\mathbf{w}) = \dim_{\Bbbk}\big(\,\Bbbk[x_{d+1}^{\pm 1}, \ldots, x_n^{\pm 1}]/(\mathrm{in}_{\mathbf{w}}(I) \cap \Bbbk[x_{d+1}^{\pm 1}, \ldots, x_n^{\pm 1}])\,\big)$$

(see [MS15, Lemma 3.44]).

We now extend the definition of initial ideals to congruences on $\tilde{S}$ and S.

Definition 3.1. Let $F = \bigoplus a_{\mathbf{u}} \odot \mathbf{x}^{\mathbf{u}} \in \tilde{S}$ and $\mathbf{w} \in \mathbb{R}^{n+1}$. The *initial form* of F with respect to $\mathbf{w}$ is the tropical polynomial in $\mathbb{B}[x_0, \ldots, x_n]$

$$\mathrm{in}_{\mathbf{w}}(F) := \bigoplus_{a_{\mathbf{u}} + \mathbf{w}\cdot\mathbf{u} = F(\mathbf{w})} \mathbf{x}^{\mathbf{u}}.$$

For $G = \bigoplus b_{\mathbf{v}} \odot \mathbf{x}^{\mathbf{v}} \in \tilde{S}$, let $\gamma = \min(F(\mathbf{w}), G(\mathbf{w}))$. The initial form of the relation $F \sim G$ with respect to $\mathbf{w}$ is the relation

$$\mathrm{in}_{\mathbf{w}}(F \sim G) \quad := \quad \bigoplus_{a_{\mathbf{u}} + \mathbf{w}\cdot\mathbf{u} = \gamma} \mathbf{x}^{\mathbf{u}} \sim \bigoplus_{b_{\mathbf{v}} + \mathbf{w}\cdot\mathbf{v} = \gamma} \mathbf{x}^{\mathbf{v}}.$$

Note that if $F(\mathbf{w}) = G(\mathbf{w})$ then this is $\mathrm{in}_{\mathbf{w}}(F) \sim \mathrm{in}_{\mathbf{w}}(G)$, but if $F(\mathbf{w}) < G(\mathbf{w})$ then this is $\mathrm{in}_{\mathbf{w}}(F) \sim \infty$.

For a congruence J on $\tilde{S}$, the *initial congruence* of J with respect to $\mathbf{w}$ is the congruence on $\mathbb{B}[x_0, \ldots, x_n]$

$$\mathrm{in}_{\mathbf{w}}(J) := \langle \mathrm{in}_{\mathbf{w}}(F \sim G) : F \sim G \in J \rangle.$$

The initial form with respect to $\mathbf{w} \in \mathbb{R}^n$ of a relation between tropical Laurent polynomials and the initial congruence of a congruence on S are defined analogously.

Example 3.2. Consider $S = \overline{\mathbb{R}}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$, and let $F = 0 \odot x \oplus 1 \odot y \oplus 2 \odot z \in S$. For $\mathbf{u} = (1, 0, 0)$ we have the relation $F \sim F_{\hat{\mathbf{u}}}$, which is $0 \odot x \oplus 1 \odot y \oplus 2 \odot z \sim 1 \odot y \oplus 2 \odot z$. If $\mathbf{w} = (2, 1, 3)$, the initial form $\mathrm{in}_{\mathbf{w}}(F \sim F_{\hat{\mathbf{u}}})$ of this relation is $x \oplus y \sim y$. For $\mathbf{w} = (1, 2, 2)$ the initial form is $x \sim \infty$. ♢

As in standard Gröbner theory, the initial congruence of a congruence generated by $\{F_\alpha \sim G_\alpha\}_{\alpha \in A}$ for some set A is not necessarily generated by $\{\mathrm{in}_{\mathbf{w}}(F_\alpha \sim G_\alpha)\}_{\alpha \in A}$. For example, for $\mathbf{w} = (0, 1, 2)$ and the congruence J on $\overline{\mathbb{R}}[x, y, z]$ generated by $\{x \sim y, x \sim z\}$, we have $y \sim z \in J$, so $y \sim \infty \in \mathrm{in}_{\mathbf{w}}(J)$. However, the initial form of both $x \sim y$ and $x \sim z$ is $x \sim \infty$, and $y \sim \infty \notin \langle x \sim \infty \rangle$.

Definition 3.1 is designed to commute with tropicalization of polynomials, as the following lemma shows.

Lemma 3.3. *For $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $\mathbf{w} \in \mathbb{R}^n$ we have*

$$\mathrm{in}_{\mathbf{w}}(\mathrm{trop}(f)) = \mathrm{trop}(\mathrm{in}_{\mathbf{w}}(f)).$$

The same holds for $f \in K[x_0, \ldots, x_n]$ and $\mathbf{w} \in \mathbb{R}^{n+1}$.

Proof. Suppose $f = \sum c_{\mathbf{u}}\mathbf{x}^{\mathbf{u}}$ with $c_{\mathbf{u}} \in K$, so $\operatorname{trop}(f) = \bigoplus \operatorname{val}(c_{\mathbf{u}}) \odot \mathbf{x}^{\mathbf{u}}$. Let $\gamma = \operatorname{trop}(f)(\mathbf{w})$. By definition, $\operatorname{in}_{\mathbf{w}}(f) = \sum_{\operatorname{val}(c_{\mathbf{u}})+\mathbf{w}\cdot\mathbf{u}=\gamma} \overline{t^{-\operatorname{val}(c_{\mathbf{u}})}c_{\mathbf{u}}}\mathbf{x}^{\mathbf{u}} \in \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Thus $\operatorname{trop}(\operatorname{in}_{\mathbf{w}}(f)) = \bigoplus_{\operatorname{val}(c_{\mathbf{u}})+\mathbf{w}\cdot\mathbf{u}=\gamma} \mathbf{x}^{\mathbf{u}} = \operatorname{in}_{\mathbf{w}}(\operatorname{trop}(f))$, as claimed. □

The first key result of this section is the following, which says that taking congruences commutes with taking initial ideals.

Proposition 3.4. *Let I be an ideal in $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then for $\mathbf{w} \in \mathbb{R}^n$ we have*
$$\operatorname{in}_{\mathbf{w}}(\mathcal{T}rop(I)) = \mathcal{T}rop(\operatorname{in}_{\mathbf{w}}(I)).$$

Proof. Fix $\mathbf{w} \in \mathbb{R}^n$. The congruence $\mathcal{T}rop(\operatorname{in}_{\mathbf{w}}(I))$ is generated by relations of the form $\operatorname{trop}(g) \sim \operatorname{trop}(g)_{\hat{\mathbf{v}}}$ for $g \in \operatorname{in}_{\mathbf{w}}(I)$ and $\mathbf{v} \in \operatorname{supp}(g)$. We first note that we can write $g = \sum \operatorname{in}_{\mathbf{w}}(f_i)$ for some $f_i \in I$ with $\operatorname{supp}(\operatorname{in}_{\mathbf{w}}(f_i)) \cap \operatorname{supp}(\operatorname{in}_{\mathbf{w}}(f_j)) = \emptyset$ if $i \neq j$. Indeed, if $g = \sum a_i \mathbf{x}^{\mathbf{u}_i} \operatorname{in}_{\mathbf{w}}(f_i)$ for $a_i \in \Bbbk$ and $f_i \in I$, then for $c_i \in R$ with $\overline{c_i} = a_i$ we have $g = \sum \operatorname{in}_{\mathbf{w}}(c_i \mathbf{x}^{\mathbf{u}_i} f_i)$, so we may assume that $\mathbf{u}_i = \mathbf{0}$ and $a_i = 1$. If the minimum in both $\operatorname{trop}(f_i)(\mathbf{w})$ and $\operatorname{trop}(f_j)(\mathbf{w})$ is achieved at the term involving $\mathbf{u}$, where the coefficient of $\mathbf{x}^{\mathbf{u}}$ in f_i is c and the coefficient in f_j is d, then $\gamma := \operatorname{trop}(f_j)(\mathbf{w}) - \operatorname{trop}(f_i)(\mathbf{w}) = \operatorname{val}(d) - \operatorname{val}(c) \in \Gamma$, and we can find $\alpha \in K$ with $\operatorname{val}(\alpha) = \operatorname{val}(d) - \operatorname{val}(c)$ and $\overline{\alpha t^{-\operatorname{val}(\alpha)}} = 1$. We then have $h = f_j + \alpha f_i \in I$, and $\operatorname{in}_{\mathbf{w}}(h) = \operatorname{in}_{\mathbf{w}}(f_i) + \operatorname{in}_{\mathbf{w}}(f_j)$. We may thus replace f_i, f_j by h, and repeat this procedure until the supports of the $\operatorname{in}_{\mathbf{w}}(f_i)$ are disjoint. Note that this implies that $\operatorname{trop}(g) = \bigoplus \operatorname{trop}(\operatorname{in}_{\mathbf{w}}(f_i))$.

Now, for $\mathbf{v} \in \operatorname{supp}(\operatorname{in}_{\mathbf{w}}(f_1))$ we can write $H = \bigoplus_{i=2}^{s}(\operatorname{trop}(\operatorname{in}_{\mathbf{w}}(f_i)))$, so $\operatorname{trop}(g) \sim \operatorname{trop}(g)_{\hat{\mathbf{v}}}$ is equal to $\operatorname{trop}(\operatorname{in}_{\mathbf{w}}(f_1)) \oplus H \sim \operatorname{trop}(\operatorname{in}_{\mathbf{w}}(f_1))_{\hat{\mathbf{v}}} \oplus H$. This shows that $\mathcal{T}rop(\operatorname{in}_{\mathbf{w}}(I))$ is generated by relations of the form $\operatorname{trop}(\operatorname{in}_{\mathbf{w}}(f)) \sim \operatorname{trop}(\operatorname{in}_{\mathbf{w}}(f))_{\hat{\mathbf{v}}}$. Since $\min(\operatorname{trop}(f)(\mathbf{w}), \operatorname{trop}(f)_{\hat{\mathbf{v}}}(\mathbf{w})) = \operatorname{trop}(f)(\mathbf{w})$, we have that $\operatorname{in}_{\mathbf{w}}(\operatorname{trop}(f) \sim \operatorname{trop}(f)_{\hat{\mathbf{v}}})$ equals $\operatorname{in}_{\mathbf{w}}(\operatorname{trop}(f)) \sim \operatorname{in}_{\mathbf{w}}(\operatorname{trop}(f))_{\hat{\mathbf{v}}}$, so by Lemma 3.3, $\operatorname{in}_{\mathbf{w}}(\operatorname{trop}(f) \sim \operatorname{trop}(f)_{\hat{\mathbf{v}}})$ is equal to $\operatorname{trop}(\operatorname{in}_{\mathbf{w}}(f)) \sim \operatorname{trop}(\operatorname{in}_{\mathbf{w}}(f))_{\hat{\mathbf{v}}}$. Note that the term $\operatorname{trop}(\operatorname{in}_{\mathbf{w}}(f))_{\hat{\mathbf{v}}}$ may equal ∞. This proves the containment $\operatorname{in}_{\mathbf{w}}(\mathcal{T}rop(I)) \supseteq \mathcal{T}rop(\operatorname{in}_{\mathbf{w}}(I))$.

For the reverse inclusion, let $(F' \sim G') = \operatorname{in}_{\mathbf{w}}(F \sim G)$ be a generator of the congruence $\operatorname{in}_{\mathbf{w}}(\mathcal{T}rop(I))$, where $F \sim G \in \mathcal{T}rop(I)$. Fix a chain $F = F_0 \sim F_1 \sim \dots \sim F_s \sim F_{s+1} = G$ in $\mathcal{T}rop(I)$ with $F_i = m_i \odot \operatorname{trop}(g_i) \oplus H_i$ and $F_{i+1} = m_i \odot \operatorname{trop}(g_i)_{\hat{\mathbf{v}}} \oplus H_i$ (or the reverse), satisfying the conditions of Proposition 2.6. In particular, we have $\gamma := \min(F(\mathbf{w}), G(\mathbf{w})) \leq F_i(\mathbf{w})$ for all i. For any $F_i = \bigoplus a_{\mathbf{u}} \odot \mathbf{x}^{\mathbf{u}}$ in this chain, define $F_i' := \bigoplus_{a_{\mathbf{u}}+\mathbf{x}\cdot\mathbf{u}=\gamma} \mathbf{x}^{\mathbf{u}}$. Note that F_i' might be equal to ∞. We claim that the chain $F' = F_0' \sim F_1' \sim \dots \sim F_s' \sim F_{s+1}' = G'$ is a chain of relations in $\mathcal{T}rop(\operatorname{in}_{\mathbf{w}}(I))$. It follows that $F' \sim G' \in \mathcal{T}rop(\operatorname{in}_{\mathbf{w}}(I))$, completing the proof.

To prove the claim, consider first the case where $F_i(\mathbf{w}) = F_{i+1}(\mathbf{w}) = \gamma$ for some i. If $m_i + \operatorname{trop}(g_i)(\mathbf{w}) > H_i(\mathbf{w}) = \gamma$ then $(F_i' \sim F_{i+1}') = (\operatorname{in}_{\mathbf{w}}(H_i) \sim \operatorname{in}_{\mathbf{w}}(H_i))$. If $m_i + \operatorname{trop}(g_i)(\mathbf{w}) = H_i(\mathbf{w}) = \gamma$ then $F_i' \sim F_{i+1}'$ equals
$$\operatorname{in}_{\mathbf{w}}(\operatorname{trop}(g_i)) \oplus \operatorname{in}_{\mathbf{w}}(H_i) \sim \operatorname{in}_{\mathbf{w}}(\operatorname{trop}(g_i))_{\hat{\mathbf{u}}} \oplus \operatorname{in}_{\mathbf{w}}(H_i),$$
where we note that $\operatorname{in}_{\mathbf{w}}(\operatorname{trop}(g_i))_{\hat{\mathbf{u}}}$ may equal ∞. If $\gamma = m_i + \operatorname{trop}(g_i)(\mathbf{w}) < H_i(\mathbf{w})$ then $F_i' \sim F_{i+1}'$ is equal to $\operatorname{in}_{\mathbf{w}}(\operatorname{trop}(g_i)) \sim \operatorname{in}_{\mathbf{w}}(\operatorname{trop}(g_i))_{\hat{\mathbf{u}}}$. In all cases, Lemma 3.3 ensures that the relation $F_i' \sim F_{i+1}'$ is in $\mathcal{T}rop(\operatorname{in}_{\mathbf{w}}(I))$. Now, suppose that $F_i(\mathbf{w}) < F_{i+1}(\mathbf{w})$. If $\gamma = F_i(\mathbf{w})$ then $\operatorname{trop}(g_i)(\mathbf{w}) < H_i(\mathbf{w})$ and $\operatorname{in}_{\mathbf{w}}(g_i)$ is a monomial. This

means that $(F'_i \sim F'_{i+1}) = (\operatorname{trop}(\operatorname{in}_{\mathbf{w}}(g_i)) \sim \infty) \in \mathcal{T}rop(\operatorname{in}_{\mathbf{w}}(I))$. Finally, if $\gamma < F_i(\mathbf{w})$ then $F'_i \sim F'_{i+1}$ is the relation $\infty \sim \infty$, which is in $\mathcal{T}rop(\operatorname{in}_{\mathbf{w}}(I))$. □

Note that the second condition in Proposition 2.6 was crucial in this proof.

We are now ready to prove Theorem 1.2. The proof requires understanding the effect of changes of coordinates on tropical varieties and congruences. The group $\mathrm{GL}(n, \mathbb{Z})$ acts on S by monomial change of coordinates. Explicitly, a matrix A sends a tropical polynomial $f(\mathbf{x}) = \bigoplus a_{\mathbf{u}} \odot \mathbf{x}^{\mathbf{u}}$ to $\bigoplus a_{\mathbf{u}} \odot \mathbf{x}^{A\mathbf{u}} = f(A^T\mathbf{x})$. We write $A \cdot f$ for this transformed polynomial. If J is a congruence on S then $A \cdot J$ is the congruence generated by $\{A \cdot f \sim A \cdot g : f \sim g \in J\}$. This action is the tropicalization of the action of $\mathrm{GL}(n, \mathbb{Z})$ on $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ that sends a monomial $\mathbf{x}^{\mathbf{u}}$ to $\mathbf{x}^{A\mathbf{u}}$. Moreover, the action commutes with tropicalization: We have $\operatorname{trop}(A \cdot f) = A \cdot \operatorname{trop}(f)$. In particular, this implies that if I is an ideal in $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ then $\operatorname{trop}(V(A \cdot I)) = A \cdot \operatorname{trop}(V(I))$; see [MS15, Corollary 3.2.13].

Proof of Theorem 1.2. Let $\mathbf{w}$ lie in the relative interior of a maximal cell σ of the tropical variety $\operatorname{trop}(V(I))$, and let $L = \operatorname{span}(\mathbf{w} - \mathbf{w}' : \mathbf{w}' \in \sigma)$. By [MS15, Lemma 3.3.6] we have $L = \operatorname{trop}(V(\operatorname{in}_{\mathbf{w}}(I)))$, so Equation (2.1) implies that L can be recovered from the congruence $\mathcal{T}rop(\operatorname{in}_{\mathbf{w}}(I)) = \operatorname{in}_{\mathbf{w}}(\mathcal{T}rop(I))$. This means that L is determined by $\operatorname{in}_{\mathbf{w}}(\mathcal{T}rop(I))$, and thus by $\mathcal{T}rop(I)$.

After a monomial change of coordinates we may assume that $L = \operatorname{span}(\mathbf{e}_1, \dots, \mathbf{e}_d)$. By [MS15, Corollary 2.4.10] the initial ideal $\operatorname{in}_{\mathbf{w}}(I)$ is homogeneous with respect to the $\mathbb{Z}^d$-grading by $\deg(x_i) = \mathbf{e}_i$ for $1 \leq i \leq d$ and $\deg(x_i) = \mathbf{0}$ otherwise, so it has a generating set $f_1, \dots, f_r$ where $f_i \in \Bbbk[x_{d+1}^{\pm 1}, \dots, x_n^{\pm 1}]$. By [MS15, Lemma 3.4.7], the multiplicity of σ equals the dimension $\dim_{\Bbbk}(R'/(\operatorname{in}_{\mathbf{w}}(I) \cap R'))$, where $R' = \Bbbk[x_{d+1}^{\pm 1}, \dots, x_n^{\pm 1}]$. Let $\operatorname{in}_{\mathbf{w}}(I)^h \subset \Bbbk[x_0, \dots, x_n]$ be the homogenization of the ideal $\operatorname{in}_{\mathbf{w}}(I) \cap \Bbbk[x_1, \dots, x_n]$. Note that since $R'/(\operatorname{in}_{\mathbf{w}}(I) \cap R')$ is zero-dimensional, the Hilbert polynomial of $\Bbbk[x_0, x_{d+1}, \dots, x_n]/(\operatorname{in}_{\mathbf{w}}(I)^h)$ is equal to the constant polynomial $\dim_{\Bbbk}(R'/(\operatorname{in}_{\mathbf{w}}(I) \cap R'))$, and thus equals $\operatorname{mult}(\mathbf{w})$.

By [GG16, Theorem 7.1.6] the Hilbert polynomial of a homogeneous ideal J can be recovered from its tropicalization $\mathcal{T}rop(J) \subset \tilde{S}$, so to show that $\operatorname{mult}(\mathbf{w})$ can be recovered from $\mathcal{T}rop(I)$ it is enough to show that $\mathcal{T}rop(\operatorname{in}_{\mathbf{w}}(I)^h)$ can be recovered from $\mathcal{T}rop(I)$. By Proposition 2.9 we have $\mathcal{T}rop(\operatorname{in}_{\mathbf{w}}(I)^h) = \mathcal{T}rop(\operatorname{in}_{\mathbf{w}}(I))^h$, and by Proposition 3.4 we have $\mathcal{T}rop(\operatorname{in}_{\mathbf{w}}(I))^h = \operatorname{in}_{\mathbf{w}}(\mathcal{T}rop(I))^h$, so the result follows. □

We can thus recover the tropical cycle from the tropical scheme. This can be considered as a tropicalization of the Hilbert-Chow morphism that takes a scheme to the underlying cycle.

4. Tropical schemes and valuated matroids

In this section we investigate in more depth the structure of the equivalence classes of $\mathcal{T}rop(I)$. We restrict our attention to the case where I is a homogeneous ideal in the polynomial ring $K[x_0, \dots, x_n]$; an understanding in this case extends to ideals in $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ using Proposition 2.9. We prove that any homogeneous tropical polynomial $F \in \tilde{S}$ has a distinguished representative in its equivalence class, and we give a computationally tractable description of it. The combinatorial machinery that naturally keeps track of the information contained in the congruence $\mathcal{T}rop(I)$ is that of *valuated matroids*.

Valuated matroids are a generalization of the notion of matroids that were introduced by Dress and Wenzel in [DW92]. Our sign convention is, however, the opposite of theirs. For basics of standard matroids, see, for example, [Oxl92].

Let E be a finite set, and let $r \in \mathbb{N}$. Denote by $\binom{E}{r}$ the collection of subsets of E of size r. A *valuated matroid* $\mathcal{M}$ on the ground set E is a function $p\colon \binom{E}{r} \to \overline{\mathbb{R}}$ satisfying the following properties.

(1) There exists $B \in \binom{E}{r}$ such that $p(B) \neq \infty$.
(2) *Tropical Plücker relations:* For every $B, B' \in \binom{E}{r}$ and every $\mathbf{u} \in B - B'$ there exists $\mathbf{v} \in B' - B$ with
$$p(B) + p(B') \geq p(B - \mathbf{u} \cup \mathbf{v}) + p(B' - \mathbf{v} \cup \mathbf{u}).$$

The support
$$\operatorname{supp}(p) := \{B \in \tbinom{E}{r} : p(B) \neq \infty\}$$
is the collection of bases of a rank r matroid on the ground set E, called the *underlying matroid* $\underline{\mathcal{M}}$ of $\mathcal{M}$. The function p is called the *basis valuation function* of $\mathcal{M}$. We consider the basis valuation functions p and $\lambda + p$ for $\lambda \in \mathbb{R}$ to be the same valuated matroid.

We denote by M_d the set of monomials of degree d in the variables $x_0, \dots, x_n$. Any homogeneous polynomial $f \in K[x_0, \dots, x_n]$ of degree d can be regarded as a linear form l_f on the K-vector-space V_d with basis M_d. Let I_d be the degree d part of the ideal I. Consider the linear subspace
$$L_d := \{\mathbf{y} \in V_d : l_f(\mathbf{y}) = 0 \text{ for all } f \in I_d\} \subset V_d.$$
Under the pairing $\langle \cdot, \cdot \rangle : K[x_0, \dots, x_n]_d \times V_d \to K$ defined by $\langle f, \mathbf{y} \rangle := l_f(\mathbf{y})$, the linear subspace L_d is orthogonal to I_d. Let $r_d = \dim(L_d)$. The linear subspace L_d determines a point in the Grassmannian $\operatorname{Gr}(r_d, V_d)$. The coordinates of this point in the Plücker embedding of $\operatorname{Gr}(r_d, V_d)$ into $\mathbb{P}^N$, where $N := \binom{|M_d|}{r_d} - 1$, are called the Plücker coordinates of L_d (and dually of I_d). They are indexed by subsets of M_d of size r_d.

Definition 4.1. The valuated matroid $\mathcal{M}(I_d)$ of I_d is the function $p_d : \binom{M_d}{r_d} \to \overline{\mathbb{R}}$ given by setting $p_d(B)$ to be the valuation of the Plücker coordinate of $L_d \in \operatorname{Gr}(r_d, \binom{n+d}{d})$ indexed by B. We denote by $\underline{\mathcal{M}}(I_d)$ the underlying matroid of $\mathcal{M}(I_d)$.

A valuated matroid that comes from taking the valuation of Plücker coordinates is called *realizable*. The function p_d is the *tropical Plücker vector* associated to the tropical linear space $\operatorname{trop}(L_d)$; it completely determines $\operatorname{trop}(L_d)$ [SS04, Theorem 3.8]. The valuated matroids that arise in the tropicalization of an ideal are all realizable, but we will not need that fact in the proofs in this section.

Usual matroids have several different "cryptomorphic" definitions, and the same is true for valuated matroids. In the underlying matroid $\underline{\mathcal{M}}(I_d)$, a subset of monomials $A \subset M_d$ is dependent if and only if there exists a polynomial $h \in I_d$ with $\operatorname{supp}(h) \subset A$.

Thus $C \subset M_d$ is a *circuit* of $\underline{\mathcal{M}}(I_d)$ if and only if $C = \operatorname{supp}(h)$ for some $h \in I_d$ of minimal support. A tropical polynomial $H \in \operatorname{trop}(I)_d$ is called a *vector* of the valuated matroid $\mathcal{M}(I_d)$. Such an H has the form $\bigoplus_{i=1}^{s} a_i \odot \operatorname{trop}(f_i)$ for some $f_i \in I_d$ and $a_i \in \mathbb{R}$. Vectors of minimal support are called *valuated circuits* of $\mathcal{M}(I_d)$. These all have the form $H = a \odot \operatorname{trop}(h)$ for some $h \in I_d$ of minimal support and $a \in \mathbb{R}$. If H and G are valuated circuits of $\mathcal{M}(I_d)$ with the same support then there exists some $a \in \mathbb{R}$ such that $H = a \odot G$. The set of vectors and the set of valuated circuits of $\mathcal{M}(I_d)$ each separately determines $\mathcal{M}(I_d)$; see [MT01, Theorem 3.3].

With these definitions in place, we can now finish the proof of Theorem 1.1. We restate it in a slightly generalized form, allowing more general projective schemes.

Theorem 4.2. *Let $Z \subset \mathbb{P}^n$ be a subscheme defined by a homogeneous ideal $I \subset K[x_0, \dots, x_n]$. Then any of the following three objects determines the others:*

1. *The congruence $\mathcal{T}rop(I)$ on $\tilde{S}$;*
2. *The ideal $\operatorname{trop}(I)$ in $\tilde{S}$;*
3. *The set of valuated matroids $\{\mathcal{M}(I_d)\}_{d \geq 0}$, where I_d is the degree d part of I.*

Thus if $Y \subset T \cong (K^*)^n$ is a subscheme given by an ideal $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and $I^h \subset K[x_0, \dots, x_n]$ is the ideal of the projective closure $\overline{Y} \subset \mathbb{P}^n$ of Y, then the ideal $\operatorname{trop}(I) \subset S$ and the set of valuated matroids $\mathcal{M}(I^h_d)$ for $d \geq 0$ determine each other.

Proof. The proof that (1) determines (2) given at the end of Section 2 included the proof for general homogeneous ideals, as we never used that I^h was a homogenization. The proof given there that (2) determines (1) is also valid for homogeneous ideals.

The elements of $\operatorname{trop}(I)_d$ are the vectors of the valuated matroid $\mathcal{M}(I_d)$, so $\operatorname{trop}(I)$ determines and is determined by the set of valuated matroids $\{\mathcal{M}(I_d)\}_{d \geq 0}$. This shows (2) $\Leftrightarrow$ (3).

When $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the ideal $\operatorname{trop}(I^h)$ in $\tilde{S}$ is the homogenization of the ideal $\operatorname{trop}(I)$ in S, and also $\operatorname{trop}(I) = \operatorname{trop}(I^h)|_{x_0=0}$. This shows that $\operatorname{trop}(I)$ determines $\operatorname{trop}(I^h)$ and conversely, so the last part follows from the first. ∎

We now investigate in more depth the structure of the equivalence classes of $\mathcal{T}rop(I)$. In what follows, for any homogeneous tropical polynomial $F \in \tilde{S}_d$ and any $\mathbf{u} \in M_d$, we denote by $F^{\mathbf{u}}$ the coefficient of F corresponding to the monomial $\mathbf{u}$. For $F, G \in \tilde{S}_d$, we say that $F \leq G$ if the inequality holds coefficient-wise, so $F^{\mathbf{u}} \leq G^{\mathbf{u}}$ for all $\mathbf{u} \in M_d$.

We will restrict our attention to the case where the subspace I_d does not contain any monomials, so the matroid $\underline{\mathcal{M}}(I_d)$ is a loopless matroid. When I does contain a monomial $g = a\mathbf{x}^{\mathbf{u}}$, the congruence $\mathcal{T}rop(I)$ contains the relation $\operatorname{trop}(g) \sim \operatorname{trop}(g)_{\hat{\mathbf{u}}} = \infty$. For any tropical polynomial $P \in \tilde{S}_d$ and any $\lambda \in \mathbb{R}$, the relation $P \oplus \lambda \odot \mathbf{x}^{\mathbf{u}} \sim P$ is then in $\mathcal{T}rop(I)$. This implies that the equivalence class of a tropical polynomial $F \in \tilde{S}_d$ does not depend on the coefficient of the monomial $\mathbf{u}$, so we do not lose information by ignoring this coefficient.

Let $F = \bigoplus_{\mathbf{u} \in M_d} F^{\mathbf{u}} \odot \mathbf{x}^{\mathbf{u}}$ be a homogeneous tropical polynomial of degree d. For any circuit $C \subset M_d$ of $\underline{\mathcal{M}}(I_d)$ and any $\mathbf{u} \in C$, let $G_{C,\mathbf{u}}$ be the valuated circuit of

$\mathcal{M}(I_d)$ satisfying $\operatorname{supp}(G_{C,\mathbf{u}}) = C$ and $G^{\mathbf{u}}_{C,\mathbf{u}} = 0$. Furthermore, let

$$\lambda_{C,\mathbf{u}} := \max_{\mathbf{v}\in C-\mathbf{u}} (F^{\mathbf{v}} - G^{\mathbf{v}}_{C,\mathbf{u}}) \in \overline{\mathbb{R}}. \tag{4.1}$$

The subtraction here is in usual arithmetic, where we follow the convention that $\infty - a = \infty$ for $a \in \mathbb{R}$. Since $\mathbf{v} \in C$ we have $G^{\mathbf{v}}_{C,\mathbf{u}} < \infty$. The assumption that $\underline{\mathcal{M}}(I_d)$ is loopless ensures that this maximum is over a nonempty set, so $\lambda_{C,\mathbf{u}} \in \overline{\mathbb{R}}$. Equivalently, $\lambda_{C,\mathbf{u}}$ satisfies

$$\lambda_{C,\mathbf{u}} = \min\left(\lambda \in \overline{\mathbb{R}} : \lambda + (G_{C,\mathbf{u}})_{\hat{\mathbf{u}}} \geq F\right). \tag{4.2}$$

We define the tropical polynomial $\pi(F) \in \tilde{S}_d$ to be the tropical sum

$$\pi(F) := F \oplus \Big(\bigoplus_{\mathbf{u}\in C\subset M_d} \lambda_{C,\mathbf{u}} \odot G_{C,\mathbf{u}} \Big)$$

where the inner sum is taken over all circuits C of $\underline{\mathcal{M}}(I_d)$ and all $\mathbf{u} \in C$. The coefficient of $\mathbf{v}$ in $\pi(F)$ is

$$\pi(F)^{\mathbf{v}} = \min\Big(F^{\mathbf{v}}\, ,\, \min_{\mathbf{v}\in C\subset M_d} (\lambda_{C,\mathbf{v}}) \Big), \tag{4.3}$$

where the inner minimum is only over those circuits C containing $\mathbf{v}$.

Example 4.3. Consider the ideal $I = \langle x + y + tz, x + y + t^2 w\rangle$ in $\mathbb{C}\{\{t\}\}[x, y, z, w]$. The underlying matroid $\underline{\mathcal{M}}(I_1)$ in degree one has ground set $M_1 = \{x, y, z, w\}$, and circuits $\{x, y, z\}$, $\{x, y, w\}$, and $\{z, w\}$. The valuated matroid $\mathcal{M}(I_1)$ has valuated circuits $x \oplus y \oplus 1 \odot z$, $x \oplus y \oplus 2 \odot w$, and $z \oplus 1 \odot w$. Consider the tropical polynomial $F = x \oplus 1 \odot y \in \overline{\mathbb{R}}[x, y, z, w]$. The polynomial $\pi(F)$ is equal to

$$\begin{aligned}\pi(F) &= F \oplus 1 \odot (x \oplus y \oplus 1 \odot z) \oplus 1 \odot (x \oplus y \oplus 2 \odot w) \oplus \infty \odot (z \oplus 1 \odot w)\\ &= x \oplus 1 \odot y \oplus 2 \odot z \oplus 3 \odot w.\end{aligned}$$

Similarly, for the tropical polynomial $F' = 2 \odot w \in S$ we have

$$\begin{aligned}\pi(F') &= F' \oplus \infty \odot (x \oplus y \oplus 1 \odot z) \oplus \infty \odot (x \oplus y \oplus 2 \odot w) \oplus 1 \odot (z \oplus 1 \odot w)\\ &= 1 \odot z \oplus 2 \odot w.\end{aligned}$$

$\diamondsuit$

The following proposition shows that $\pi(F)$ is the coefficient-wise smallest tropical polynomial in the equivalence class of F in $\mathcal{T}rop(I)$. It is thus a distinguished representative of the equivalence class.

Proposition 4.4. *The map $\pi : \tilde{S}_d \to \tilde{S}_d$ satisfies the following properties:*

(a) $\pi(F) \leq F$.
(b) $\pi(\pi(F)) = \pi(F)$.
(c) $F \sim \pi(F) \in \mathcal{T}rop(I)$.
(d) $F \sim F' \in \mathcal{T}rop(I) \iff \pi(F) = \pi(F')$.

In the proof of Proposition 4.4 we will make use of the following facts about valuated circuits:

(1) If H is a vector of $\mathcal{M}(I_d)$ with $\mathbf{u} \in \operatorname{supp}(H)$ then there is a valuated circuit G with $G^{\mathbf{u}} = H^{\mathbf{u}}$ and $G \geq H$.

(2) If H is a vector and G is a valuated circuit of $\mathcal{M}(I_d)$ with $H^{\mathbf{u}} = G^{\mathbf{u}} < \infty$ and $H^{\mathbf{v}} > G^{\mathbf{v}}$, then there is a valuated circuit G' of $\mathcal{M}(I_d)$ with $G' \geq \min(H, G)$, $G'^{\mathbf{v}} = G^{\mathbf{v}}$, and $G'^{\mathbf{u}} = \infty$.

Fact (1) follows from Theorems 3.4 and 3.5 of [MT01] and the definition given there of the function $\phi_{\mathcal{X}\to\mathcal{V}}(\mathcal{X})$. Fact (2) is a combination of Fact (1) and the valuated circuit elimination axiom [MT01, Theorem 3.1 (VCE)].

Proof of Proposition 4.4. Property (a) follows directly from the definition, since F is a tropical summand of $\pi(F)$. Property (b) follows from properties (c) and (d), which we now prove. In order to show that Property (c) holds, fix an enumeration $\{(\mathbf{u}_1, C_1), \ldots, (\mathbf{u}_s, C_s)\}$ of the set $\{(\mathbf{u}, C) : C$ is a circuit of $\underline{\mathcal{M}}(I_d)$ and $\mathbf{u} \in C\}$. For $0 \leq i \leq s$, set

$$H_i := F \oplus \Big(\bigoplus_{1\leq j\leq i} \lambda_{C_j,\mathbf{u}_j} \odot G_{C_j,\mathbf{u}_j} \Big)$$

so that $H_0 = F$ and $H_s = \pi(F)$. By Equation (4.2), for any i we have $H_{i-1} \leq F \leq \lambda_{C_i,\mathbf{u}_i} \odot (G_{C_i,\mathbf{u}_i})_{\hat{\mathbf{u}}_i}$. Since $\mathcal{T}rop(I)$ is a congruence, the relation

$$H_{i-1} = H_{i-1} \oplus \lambda_{C_i,\mathbf{u}_i} \odot (G_{C_i,\mathbf{u}_i})_{\hat{\mathbf{u}}_i} \sim H_{i-1} \oplus \lambda_{C_i,\mathbf{u}_i} \odot G_{C_i,\mathbf{u}_i} = H_i$$

is in $\mathcal{T}rop(I)$. The result follows from transitivity.

We now prove Property (d). If $\pi(F) = \pi(F')$ then by Property (c) we have $F \sim \pi(F) = \pi(F') \sim F'$, so $F \sim F'$. In order to prove the converse statement, by Lemma 2.4 it is enough to show that $\pi(H \oplus P) = \pi(H_{\hat{\mathbf{u}}} \oplus P)$ for any vector H of $\mathcal{M}(I_d)$, $\mathbf{u} \in \operatorname{supp}(H)$, and $P \in \tilde{S}_d$. Set

$$F := H \oplus P \quad \text{ and } \quad F' := H_{\hat{\mathbf{u}}} \oplus P.$$

Note that F and F' can only differ in the coefficient corresponding to the monomial $\mathbf{u}$. We will assume that $F^{\mathbf{u}} = H^{\mathbf{u}} < P^{\mathbf{u}}$, as otherwise $F = F'$.

For any circuit C of $\underline{\mathcal{M}}(I_d)$ and any $\mathbf{u}' \in C$, let $\lambda_{C,\mathbf{u}'} \in \overline{\mathbb{R}}$ be as in Equation (4.1). Let $\lambda'_{C,\mathbf{u}'}$ be defined analogously for the tropical polynomial F'. Since $F \leq F'$, we have $\lambda_{C,\mathbf{u}'} \leq \lambda'_{C,\mathbf{u}'}$. It follows that $\pi(F) \leq \pi(F')$. Since $F^{\mathbf{v}} = F'^{\mathbf{v}}$ for $\mathbf{v} \neq \mathbf{u}$, we see from Equation (4.1) that $\lambda_{C,\mathbf{u}} = \lambda'_{C,\mathbf{u}}$. Thus Equation (4.3) implies that $\pi(F)^{\mathbf{u}} = \pi(F')^{\mathbf{u}}$.

Suppose that $\pi(F)^{\mathbf{v}} < \pi(F')^{\mathbf{v}}$ for some $\mathbf{v} \neq \mathbf{u}$. By Equation (4.3), there must be a circuit C of $\underline{\mathcal{M}}(I_d)$ with $\mathbf{v} \in C$ and both $\lambda_{C,\mathbf{v}} < \pi(F')^{\mathbf{v}} \leq \lambda'_{C,\mathbf{v}}$ and $\lambda_{C,\mathbf{v}} < F^{\mathbf{v}} = F'^{\mathbf{v}} \leq H^{\mathbf{v}}$. The maximum in Equation (4.1) must then be achieved at the coefficient of $\mathbf{u}$, as this is the only coefficient for which F and F' differ, so $\lambda_{C,\mathbf{v}} = F^{\mathbf{u}} - G^{\mathbf{u}}_{C,\mathbf{v}}$. Note that this implies in particular that $\mathbf{u} \in C$. By Fact (2) applied to the vector H and the valuated circuit $\lambda_{C,\mathbf{v}} \odot G_{C,\mathbf{v}}$, there is a valuated circuit G' of support C' with $\mathbf{u} \notin C'$, $G'^{\mathbf{v}} = \lambda_{C,\mathbf{v}}$, and $G' \geq \lambda_{C,\mathbf{v}} \odot G_{C,\mathbf{v}} \oplus H$. The valuated circuit G' must then be equal to $\lambda_{C,\mathbf{v}} \odot G_{C',\mathbf{v}}$. We now have

$$\lambda_{C,\mathbf{v}} \odot (G_{C',\mathbf{v}})_{\hat{\mathbf{v}}} = G'_{\hat{\mathbf{v}}} \geq \lambda_{C,\mathbf{v}} \odot (G_{C,\mathbf{v}})_{\hat{\mathbf{v}}} \oplus H_{\hat{\mathbf{v}}} \geq F,$$

and thus in view of Equation (4.2), $\lambda_{C',\mathbf{v}} \leq \lambda_{C,\mathbf{v}}$. Since $\mathbf{u} \notin C'$, we have $\lambda_{C',\mathbf{v}} = \lambda'_{C',\mathbf{v}}$ by Equation (4.1), which contradicts $\lambda_{C,\mathbf{v}} < \pi(F')^{\mathbf{v}} \leq \lambda'_{C',\mathbf{v}}$. We thus conclude that $\pi(F)^{\mathbf{v}} = \pi(F')^{\mathbf{v}}$ for all $\mathbf{v}$. $\square$

Remark 4.5. Note that we have $\lambda_{C,\mathbf{u}} < \infty$ if and only if $C - \mathbf{u} \subset \operatorname{supp}(F)$. It follows that $\operatorname{supp}(\pi(F))$ equals the closure E_F of the set $\operatorname{supp}(F)$ in the matroid $\underline{\mathcal{M}}(I_d)$. In fact, we may regard $\pi(F)$ as being the "valuated closure" of F in the valuated matroid $\mathcal{M}(I_d)$. The definition of π makes sense for any valuated matroid $\mathcal{M}$, and its properties stated in Proposition 4.4 remain valid in this more general setup. It would be interesting to develop a set of cryptomorphic axioms for valuated matroids from this "valuated closure" perspective.

Remark 4.6. The map π can also be thought of as a tropical projection, in the following sense. One can extend the function p_d to all subsets of M_d, obtaining in this way a valuation function for all independent subsets of $\underline{\mathcal{M}}(I_d)$ [Mur97]. Concretely, for any $A \subset M_d$ define

$$p_d(A) := \min\{p_d(B) : A \subset B \in \tbinom{M_d}{r_d}\},$$

with the convention that $p_d(A) = \infty$ if the corresponding set is empty. We have $p_d(A) \neq \infty$ if and only if A is an independent set of $\underline{\mathcal{M}}(I_d)$. Given any subset $E \subset M_d$, the restriction of the function p_d to the maximal independent subsets of E gives rise to a valuated matroid on the set E, called the restriction $\mathcal{M}(I_d)|E$ of $\mathcal{M}(I_d)$ to E.

Now, suppose F is a homogeneous tropical polynomial of degree d. Let E_F be the closure of $\operatorname{supp}(F)$ in the matroid $\underline{\mathcal{M}}(I_d)$. If $\operatorname{supp}(F) = E_F$, it follows from Equation (4.3) and [Cor13, Section 4] that $\pi(F)$ is the tropical projection of $F \in \overline{\mathbb{R}}^{E_F}$ onto the tropical linear space in $\overline{\mathbb{R}}^{E_F}$ corresponding to the valuated matroid $\underline{\mathcal{M}}(I_d)|E_F$, *but taking tropical sum to be* max *instead of of* min. If $\operatorname{supp}(F) \subsetneq E_F$ then we have to be more careful: The tropical polynomial $\pi(F)$ is the tropical projection of F after substituting the coefficients corresponding to monomials in $E_F - \operatorname{supp}(F)$ by large enough real numbers.

Tropical projections onto tropical linear spaces have been studied in [Ard04, Cor13, Rin13]. Using those results one can obtain a description of $\pi(F)$ amenable to computational purposes, as we now describe. For any basis B of E_F (i.e., a maximal independent set contained in E_F), define

$$w_F(B) := p_d(B) + \textstyle\sum_{\mathbf{u} \in B} F^{\mathbf{u}} \in \overline{\mathbb{R}}.$$

Let B_F be a basis of E_F such that $w_F(B_F)$ is minimal among all bases B_F of E_F. Note that the value of $w_F(B_F)$ is finite, so $B_F \subset \operatorname{supp}(F)$. For any $\mathbf{u} \in E_F - B_F$ there exists a unique circuit $C(B_F, \mathbf{u})$ of $\underline{\mathcal{M}}(I_d)$ contained in $B_F \cup \mathbf{u}$, called the *fundamental circuit* of $\mathbf{u}$ over B_F. It is equal to

$$C(B_F, \mathbf{u}) = \{\mathbf{v} \in B_F : B_F \cup \mathbf{u} - \mathbf{v} \text{ is independent in } \underline{\mathcal{M}}(I_d)\} \cup \mathbf{u}.$$

With this notation in place, [Cor13, Section 4, Proposition 5] implies that the coefficients of $\pi(F)$ are given by

$$\pi(F)^{\mathbf{u}} = \begin{cases} F^{\mathbf{u}} & \text{if } \mathbf{u} \in B_F, \\ \displaystyle\max_{\mathbf{v} \in C(B_F,\mathbf{u}) - \mathbf{u}} (F^{\mathbf{v}} - p_d(B_F \cup \mathbf{u} - \mathbf{v}) + p_d(B_F)) & \text{if } \mathbf{u} \in E_F - B_F, \\ \infty & \text{if } \mathbf{u} \notin E_F. \end{cases}$$

The computation of the coefficients $\pi(F)^{\mathbf{u}}$ using this description involves computing a maximum over only one circuit of $\underline{\mathcal{M}}(I_d)$. This makes it computationally much simpler than formula (4.3), assuming that we know the function p_d.

References

[Ard04] Federico Ardila, *Subdominant matroid ultrametrics*, Ann. Comb. **8** (2004), no. 4, 379–389.

[Cor13] Eduardo Corel, *Gérard-Levelt membranes*, J. Algebraic Combin. **37** (2013), no. 4, 757-776.

[DW92] Andreas Dress and Walter Wenzel, *Valuated matroids*, Adv. Math. **93** (1992), no. 2, 214–250.

[GG16] Jeffrey Giansiracusa and Noah Giansiracusa, *Equations of tropical varieties*, Duke Math. J. **165** (2016), no. 18, 3379–3433.

[KM14] Thomas Kahle and Ezra Miller, *Decompositions of commutative monoid congruences and binomial ideals*, Algebra Number Theory **8** (2014), no. 6, 1297–1364.

[KP11] Eric Katz and Sam Payne, *Realization spaces for tropical fans*, Combinatorial aspects of commutative algebra and algebraic geometry, Abel Symp., vol. 6, Springer, Berlin, 2011, pp. 73–88.

[MS15] Diane Maclagan and Bernd Sturmfels, *Introduction to tropical geometry*, Graduate Studies in Mathematics, vol. 161, American Mathematical Society, Providence, RI, 2015.

[Mur97] Kazuo Murota, *Matroid valuation on independent sets*, J. Combin. Theory Ser. B **69** (1997), no. 1, 59–78.

[MT01] Kazuo Murota and Akihisa Tamura, *On circuit valuation of matroids*, Adv. in Appl. Math. **26** (2001), no. 3, 192–225.

[Oxl92] James G. Oxley, *Matroid theory*, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1992.

[Rin12] Felipe Rincón, *Isotropical linear spaces and valuated Delta-matroids*, J. Combin. Theory Ser. A **119** (2012), no. 1, 14–32.

[Rin13] ______, *Local tropical linear spaces*, Discrete Comput. Geom. **50** (2013), no. 3, 700–713.

[SS04] David Speyer and Bernd Sturmfels, *The tropical Grassmannian*, Adv. Geom. **4** (2004), no. 3, 389–411.

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